THEORY OF
IMPULSIVE
DIFFERENTIAL
EQUATIONS

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PREFACE

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. Thus impulsive differential equations, that is, differential equations involving impulse effects, appear as a natural description of observed evolution phenomena of several real world problems. Let us describe Kruger-Thiemer model for drug distribution to show how impulses occur naturally. Consider a simple two compartmental model for drug distribution in the human body which is proposed by Kruger-Thiemer. It is assumed that the drug, which is administered orally, is first dissolved into the gastro-intestinal tract. The drug is then absorbed into the so-called apparent volume of distribution (a lumped compartment which accounts for blood, muscle, tissue, etc.), and finally is eliminated from the system by the kidneys. Let \( x(t) \) and \( y(t) \) denote the amounts of drug at time \( t \) in the gastro-intestinal tract and apparent volume of distribution respectively, and let \( k_1 \) and \( k_2 \) be relevant rate constants. The dynamical system of this model is then

\[
\begin{align*}
    x' &= -k_1 x, \\
    y' &= -k_2 y + k_1 x.
\end{align*}
\]

We now postulate that at the moments of time

\[ 0 < t_1 < t_2 < \cdots < t_N < T, \]

the drug is ingested in amounts

\[ \delta_0, \delta_1, \delta_2, \cdots, \delta_N, \]
so that we have

\[
\begin{align*}
\text{(b)} \quad x(t_i^+) &= x(t_i^-) + \delta_i, \quad i = 1, 2, \ldots, N, \\
y(t_i^+) &= y(t_i^-), \\
x(0) &= \delta_0, \quad y(0) = 0.
\end{align*}
\]

To achieve the desired therapeutic effect, it is required that the amount of drug in the apparent volume of distribution never goes below a constant level or plateau during the time interval. Finally, take the biological cost function

\[f(\delta) = \frac{1}{2} \sum_{i=0}^{N} \delta_i^2\]

both to minimize side effects and the cost of the drug. The problem is to find \(\inf_{\delta \geq 0} f(\delta)\) subject to (a) and (b). Clearly (a) and (b) represent a simple impulsive differential system.

The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects. For example, initial value problems of such equations may not, in general, possess any solutions at all even when the corresponding differential equation is smooth enough; fundamental properties such as continuous dependence relative to initial data may be violated, and qualitative properties like stability may need a suitable new interpretation. Moreover, a simple impulsive differential equation may exhibit several new phenomenon such as rhythmical beating, merging of solutions, and noncontinuability of solutions. Consequently, the theory of impulsive differential equations is interesting in itself and it is easy to see that it will assume greater importance in the near future since the application of the theory to various fields is also increasing. Thus there is every reason for studying the theory of impulsive differential equations as a well deserved discipline.

The present book offers a systematic treatment of the theory of impulsive differential equations and it is divided into four chapters. Chapter 1
introduces the impulsive evolution processes, presents preliminary results and offers several examples for motivation. In Chapter 2, we consider fundamental properties of solutions, develop variation of parameters formulae, discuss the method of upper and lower solutions including monotone iterative technique and prove simple stability criteria demonstrating how the impulses effect stability behaviour of solutions. Chapter 3 is devoted to the investigation of stability theory by means of discontinuous Lyapunov functions and theory of impulsive differential inequalities. We also develop stability theory in terms of two measures which unifies several known stability concepts, discuss the method of vector Lyapunov functions, and illustrate how, in general, stability consideration may need a different outlook. Finally, in Chapter 4, we consider a variety of results dealing with different aspects of impulsive systems such as singularly perturbed systems, systems with variable structure, integro-differential systems, boundary value problems for second order differential systems, delay-differential equations and dynamical systems so as to pave the way for further investigation.

The book can be used as a textbook at the graduate level and a reference book by several disciplines.

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CHAPTER 1

1.0 INTRODUCTION.

This chapter introduces the evolution processes that are under the influence of impulsive actions and discusses certain preliminary results that form the basis of the remaining chapters.

We begin Section 1.1 by describing a set of relations which characterize an evolution process subject to impulsive effects. We then present three typical types of impulsive differential systems that are of interest and offer simple examples to demonstrate how the impulsive actions influence the behaviour of solutions and cause interesting new phenomena.

Section 1.2 is devoted to local existence and continuation results. Since an impulsive differential equation may not, in general, have any solution at all even when the corresponding differential equation is smooth enough, we give simple sets of conditions that guarantee local existence and continuation. Also, the solutions of impulsive differential systems may encounter a given surface (a threshold or a barrier) finite or infinite number of times experiencing rhythmical beating which creates problems in studying qualitative properties of solutions. In Section 1.3, we present various sufficient conditions for the presence or absence of such a phenomena.

Theory of impulsive differential inequalities which plays an important role in the investigation of qualitative theory is developed in Section 1.4, while the theory of impulsive integral inequalities forms the content of Section 1.5.

Since each solution of an impulsive differential system may experience shocks at different moments of time, when we wish to estimate the difference of
any two solutions, for example, we need the theory of differential inequalities of
different nature than the one considered in Section 1.4. Accordingly, Section 1.6
initiates such a theory of impulsive differential inequalities.

1.1 DESCRIPTION OF SYSTEMS WITH IMPULSES.

Let us consider an evolution process described by
(i) a system of differential equations

\[ x' = f(t, x), \quad t = \frac{d}{dt}, \]

where \( f: \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^n \), \( \Omega \subset \mathbb{R}^n \) is an open set, \( \mathbb{R}^n \), the n-dimensional
euclidean space and \( \mathbb{R}_+ \), the nonnegative real line;
(ii) the sets \( M(t), N(t) \subset \Omega \) for each \( t \in \mathbb{R}_+ \); and
(iii) the operator \( A(t): M(t) \rightarrow N(t) \) for each \( t \in \mathbb{R}_+ \).

Let \( x(t) = x(t, t_0, x_0) \) be any solution of (1.1.1) starting at \( (t_0, x_0) \).
The evolution process behaves as follows: the point \( P_t = (t, x(t)) \) begins its
motion from the initial point \( P_{t_0} = (t_0, x_0) \) and moves along the curve
\( \{(t, x):t \geq t_0, x = x(t)\} \) until the time \( t_1 > t_0 \) at which the point \( P_t \) meets the set
\( M(t) \). At \( t = t_1 \), the operator \( A(t) \) transfers the point \( P_{t_1} = (t_1, x(t_1)) \) into
\( P_{t_1}^+ = (t_1, x_1^+) \in N(t_1) \), where \( x_1^+ = A(t_1)x(t_1) \). Then the point \( P_t \) continues
to move further along the curve with \( x(t) = x(t, t_1, x_1^+) \) as the solution of (1.1.1)
starting at \( P_{t_1} = (t_1, x_1^+) \) until it hits the set \( M(t) \) at the next moment \( t_2 > t_1 \).

Then, once again the point \( P_{t_2} = (t_2, x(t_2)) \) is transferred to the point
\( P_{t_2}^+ = (t_2, x_2^+) \in N(t_2) \) where \( x_2^+ = A(t_2)x(t_2) \). As before, the point \( P_t \) continues
to move forward with \( x(t) = x(t, t_2, x_2^+) \) as the
solution of (1.1.1) starting at \( (t_2, x_2^+) \). Thus the evolution process continues
forward as long as the solution of (1.1.1) exists.
We shall call the set of relations (i), (ii) and (iii) that characterize the above mentioned evolution process an impulsive differential system, the curve which is described by the point \( P_t \) the integral curve and the function that defines the integral curve a solution of the impulsive differential system.

A solution of an impulsive differential system may be (a) a continuous function, if the integral curve does not intersect the set \( M(t) \) or hits it at the fixed points of the operator \( A(t) \); (b) a piecewise continuous function having finite number of discontinuities of the first kind if the integral curve meets \( M(t) \) at a finite number of points which are not the fixed points of the operator \( A(t) \); (c) a piecewise continuous function having a countable number of discontinuities of the first kind if the integral curve encounters the set \( M(t) \) at a countable number of points that are not the fixed points of the operator \( A(t) \).

The moments \( t_k \) at which the point \( P_t \) hits the set \( M(t) \) are called moments of impulsive effect. We shall assume that the solutions \( x(t) \) of the impulsive differential system is left continuous at \( t_{k-} \), \( k=1,2,\cdots, \) that is,

\[
x(t_{k-}) = \lim_{h \to 0^+} x(t_k-h) = x(t_k).
\]

The freedom we have in the choice of the set of relations (i), (ii) and (iii) that describe an impulsive differential system gives rise to several types of systems. Let us discuss the following typical impulsive differential systems that are of interest.

I. Systems with impulses at fixed times.

Let the set \( M(t) \) represent a sequence of planes \( t=t_k \) where \( \{t_k\} \) is a sequence of times such that \( t_k \to \infty \) as \( k \to \infty \). Let us define the operator \( A(t) \) for \( t=t_k \) only so that the sequence of operators \( \{A(k)\} \) is given by

\[
A(k): \Omega \to \Omega, \ x \to A(t)x = x + I_k(x),
\]

where \( I_k: \Omega \to \Omega \). As a result, the set \( N(t) \) is also defined for \( t=t_k \) and
therefore \( N(k) = A(k)M(k) \). With this choice of \( M(k) \), \( N(k) \) and \( A(k) \), a mathematical model of a simple impulsive differential system in which impulses occur at fixed times may be described by

\[
(1.1.2) \quad \begin{cases} 
  x' = f(t, x), & t \neq t_k, \ k = 1, 2, \ldots, \\
  \Delta x = I_k(x), & t = t_k,
\end{cases}
\]

where for \( t = t_k \), \( \Delta x(t_k) = x(t_k^+) - x(t_k) \) and \( x(t_k^+) = \lim_{h \to 0^+} x(t_k + h) \). We see immediately that any solution \( x(t) \) of (1.1.2) satisfies

1. \( x'(t) = f(t, x(t)), \ t \in (t_k, t_{k+1}], \) and
2. \( \Delta x(t_k) = I_k(x(t_k)), \ t = t_k, \ k = 1, 2, \ldots. \)

The behavior of solutions is influenced by the impulsive effect. The following simple examples show that the continuability of solutions is affected by the nature of impulsive action.

Example 1.1.1. Consider the impulsive differential equation

\[
(1.1.2) \quad \begin{cases} 
  x' = 0, & t \neq k, \ k = 1, 2, \ldots, \\
  \Delta x = \frac{1}{x-1}, & t = k,
\end{cases}
\]

whose solution \( x(t) \) starting at \((0,1)\) is defined only for \( 0 \leq t \leq 1 \). This solution cannot be continued for \( t > 1 \), since the function \( I_k(x) = \frac{1}{x-1} \) is not defined for \( x = 1 \). Of course, the solutions \( x(t) \) of the differential equation \( x' = 0 \) are trivially continuable for all \( t \).
Example 1.1.2. Consider, the impulsive differential equation

\[
\begin{align*}
x' &= 1 + x^2, \quad t \neq \frac{k\pi}{4}, \quad k = 1, 2, \ldots, \\
\Delta x &= -1, \quad t = \frac{k\pi}{4}.
\end{align*}
\]

The solution \( x(t) \) with \( x(0) = 0 \) is continuable for all \( t \geq 0 \). In fact, we have \( x(t) = \tan \left( t - \frac{k\pi}{4} \right) \), \( t \in \left( \frac{k\pi}{4}, \frac{(k+1)\pi}{4} \right] \) which is periodic with period \( \frac{\pi}{4} \). However, the corresponding differential equation has the solution \( x(t) = \tan t \) whose interval of existence is \([0, \frac{\pi}{2})\) since \( \lim_{t \to \frac{\pi}{2}^-} x(t) = \infty \).

II. Systems with impulses at variable times.

Let \( \{S_k\} \) be a sequence of surfaces given by \( S_k : t = \tau_k(x) \), \( k = 1, 2, \ldots \), such that \( \tau_k(x) < \tau_{k+1}(x) \) and \( \lim_{k \to \infty} \tau_k(x) = \infty \). Then we have the following impulsive differential system

\[
\begin{align*}
x' &= f(t, x), \quad t \neq \tau_k(x), \\
\Delta x &= I_k(x), \quad t = \tau_k(x), \quad k = 1, 2, \ldots.
\end{align*}
\]

Systems with variable moments of impulsive effect such as (1.1.3) offer more difficult problems compared to the systems with fixed moments of impulsive effect. For example, note that the moments of impulsive effect for the system (1.1.3) depend on the solutions i.e., \( t_k = \tau_k(x(t_k)) \), for each \( k \). Thus, solutions starting at different points will have different points of discontinuity. Also, a solution may hit the same surface \( t = \tau_k(x) \) several times and we shall call such a behaviour "pulse phenomenon". In addition, different solutions may coincide after some time and behave as a single solution thereafter. This phenomenon is called "confluence".
The following example illustrates the several situations that may arise.

**Example 1.1.3.** Consider the impulsive differential equation
\[ x' = 0, \ t \neq \tau_k(x), \ t \geq 0, \]
\[ \Delta x = x^2 \ sgn \ x - x, \ t = \tau_k(x), \ k = 0, 1, 2, \ldots, \]
where \( \tau_k(x) = x + 6k \) for \( |x| < 3 \) describe the surfaces \( S_k : t = \tau_k(x) \). We observe the following: The solutions \( x(t) \) with initial condition \( x(0) = x_0, \ |x_0| \geq 3 \)
are free from impulse effect since they do not intersect the surfaces \( S_k \). The solutions \( x(t) \) that start at the points \((0, x_0), 1 < x_0 < 3\) undergo impulsive effect a finite number of times. For example, consider the solution \( x(t) \) with \( x(0) = \frac{1}{4} \) which hits the surface \( S_0 \) three times and does not encounter any surface \( S_k \) beyond the time \( t_3 = 2 \). If the initial point \( x_0 \) is such that \( 0 < x_0 < 1 \), then the solution \( x(t) \) meets the surfaces \( S_k \) at an infinite number of times \( t_k \) and we have \( t_k \to \infty \) as \( k \to \infty \) as well as
\[ \lim_{k \to \infty} x(t_k) = 0. \]
On the other hand, the solution \( x(t) \) with \( -1 < x_0 < 0 \) encounters \( S_k \) an infinite number of times \( t_k \), but in this case \( \lim_{k \to \infty} t_k = 6 \) and \( \lim_{k \to \infty} x(t_k) = 0. \) However, the solutions starting at the points \((0, 0), (0, 1) \) and \((0, -1) \) hit the surfaces \( S_k \) at times \( t_k \) which are the fixed points of the operator \( A(t) = x^2 sgn \ x \) and for this reason, there is no impulsive effect. Finally, the solutions that start at \( (0, \frac{1}{2^4}) \) and \( (0, 4) \) unite for \( t \geq 2 \) and thus exhibit the phenomenon of confluence.

Impulsive differential systems more general than (1.1.3) may be of the form
\[(1.1.4) \quad \begin{cases} x' = f(t,x), \ h(t,x) \neq 0, \\ \Delta x = I_0(t,x), \ h(t,x) = 0. \end{cases} \]

If \( h(t,x) = 0 \) has a countable number of roots \( t = \tau_k(x) \) for each \( x \), satisfying necessary assumptions, then defining \( I_k(x) = I_0(\tau_k(x), x) \), \((1.1.4)\) reduces to \((1.1.3)\). If on the other hand, \( h(t,x) = 0 \) has the roots \( [(t_k, x) : x \in M(t_k)] \) such that \( \lim_{k \to \infty} t_k = \infty \), then \((1.1.4)\) reduces to \((1.1.2)\).

III. Autonomous systems with impulses.

Let the sets \( M(t) \equiv M \), \( N(t) \equiv N \) and the operator \( A(t) \equiv A \) be independent of \( t \) and let \( A : M \to N \) be defined by \( Ax = x + I(x) \), where \( I : \Omega \to \Omega \). Consider the autonomous impulsive differential system

\[(1.1.5) \quad \begin{cases} x' = f(x), \ x \notin M, \\ \Delta x = I(x), \ x \in M. \end{cases} \]

When any solution \( x(t) = x(t,0,x_0) \) hits the set \( M \) at some time \( t \), the operator \( A \) instantly transfers the point \( x(t) \in M \) into the point \( y(t) = x(t) + I(x(t)) \in N \). Since \((1.1.5)\) is autonomous, the motion of the point \( x(t) \) will be considered in \( \Omega \), along the trajectories of the system \((1.1.5)\).

The following examples show several interesting possibilities.

Example 1.1.4. Consider the impulsive differential system in \( \mathbb{R}^2 \) given by

\[
\begin{cases} x'_1 = -x_1, \ x'_2 = -\alpha x_2, \ \alpha > 0, \ x \notin M, \\ A : M \to N, \end{cases}
\]

where the sets \( M, N \subset \mathbb{R}^2 \) are defined by \( M = \{x \in \mathbb{R}^2 : 5x_1^2 + x_2^2 = 8\} \), \( N = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 4\} \) and \( A \) assigns to every point \( x \in M \) a point \( y \in N \) which is on
the ray joining \( x \) to the origin in \( \mathbb{R}^2 \). We shall discuss the motion of this system along the trajectories \( x_2 = c|\, x_1 |^\alpha \). We observe that every motion starting in the region \( 5x_1^2 + x_2^2 < 8 \) tends to the stationary point \( \{0\} \) and is free from impulsive effects, whereas every motion starting from the region \( 5x_1^2 + x_2^2 > 8 \) undergoes impulsive effects.

Let us first discuss the case \( 0 < \alpha < 1 \). Because of symmetry, it is enough to consider the region \( 5x_1^2 + x_2^2 > 8, \ x_1 \geq 0, x_2 \geq 0 \). Every motion that begins in the region with \( 0 \leq x_1 \leq 1 \) hits \( M \) once and then tends to the origin, while the motions starting at the points on the curve \( x_2 = \sqrt{3} \ x_1^\alpha \) which implies \( x_1 \geq 1 \) since \( 5x_1^2 + x_2^2 > 8 \), approach the origin along this curve. The solution of such a motion is continuous since it meets \( M \) at \( (1, \sqrt{3}) \) which is a fixed point of the operator \( A \). Every motion whose trajectory hits \( M \) at a point with \( 1 < x_1 < \sqrt{\frac{8}{5}} \) experiences impulses infinite number of times and then approaches the point \( (1, \sqrt{3}) \), since \( \alpha < 1 \) implies \( e^{-(t-t_0)} = e^{-\alpha(t-t_0)} \), which has the property of attracting all the motions beginning in the region \( 5x_1^2 + x_2^2 > 8, \ x_1 > 1 \) and \( 0 < x_2 < \sqrt{3} \ x_1^\alpha \). Finally, the motions starting at the points \( x_1 > \sqrt{\frac{8}{5}}, x_2 = 0 \) become periodic when they first encounter \( M \) at the point \( (\sqrt{\frac{8}{5}}, 0) \) and then the motion takes place along the segment \( \sqrt{\frac{8}{5}} \leq x_1 < 2, \ x_2 = 0 \) with periodic jumps from \( (\sqrt{\frac{8}{5}}, 0) \) to \( (2, 0) \) and period \( \frac{1}{2} \ln \frac{5}{2} \).

Let us now discuss the other case \( \alpha > 1 \). Every motion beginning from the region \( 5x_1^2 + x_2^2 < 8 \) tends to the origin and is free from impulsive effects. The motions starting in the region \( 5x_1^2 + x_2^2 > 8 \) are divided into three groups, namely,
(a) motions starting on the curve $|x_2| = \sqrt{3} |x_2|^\alpha$ have no impulsive effect;

(b) motions starting from the region $|x_2| > \sqrt{3} |x_1|^\alpha$ hit the set $M$ once;

(c) motions that begin in the region $|x_2| < \sqrt{3} |x_1|^\alpha$ encounter the set $M$ an infinite number of times.

Every motion that starts at the points such that

$$\sqrt{\frac{8}{5}} < |x_1| < 2, \ x_2 = 0$$

is periodic with period $\frac{1}{2} \ln \frac{5}{2}$. The segment

$$\sqrt{\frac{8}{5}} < x_1 \leq 2, \ x_2 = 0,$$

attracts other trajectories and the domain of attraction is the set $\{x \in \mathbb{R}^2 : |x_2|(< \sqrt{3} x_1^\alpha, \ x_1 > 0, \ 5x_1^2 + x_2^2 > 8\}$. Similarly, the domain of attraction of the segment $-2 < x_1 \leq -\sqrt{\frac{8}{5}}, \ x_2 = 0$ is the set

$$\{x \in \mathbb{R}^2 : |x_2|(< \sqrt{3} x_1^\alpha, \ x_1 < 0, \ 5x_1^2 + x_2^2 > 8\}.$$

Finally, in the case $\alpha = 1$, the character of the motion changes only in the region $\{|x_2|(< \sqrt{3} |x_1|, \ 5x_1^2 + x_2^2 > 8\}$, where all the motions become periodic and move along the ray $x_2 = cx_1$ between $M$ and $N$.

Example 1.1.5. Consider the impulsive differential system in $\mathbb{R}^2$ given by

$$\begin{align*}
x'_1 &= \alpha x_1 - \beta x_2, \quad x'_2 = \beta x_1 + \alpha x_2, \quad \alpha < 0, \ \beta > 0, \\
A: M \rightarrow N,
\end{align*}$$

where the sets $M$, $N$ are defined by $M = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = \gamma_1^2\}$ and $N = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = \gamma_2^2\}$, $\gamma_2 > \gamma_1$, such that for any point $x \in M$ there corresponds $Ax = y \in N$, lying on the same ray from the origin. Using polar
coordinates \( x_1 = \rho \cos \theta, x_2 = \rho \sin \theta \), the system (1.1.6) reduces to

\[
\begin{align*}
\frac{d\rho}{dt} &= \alpha \rho, \quad \frac{d\theta}{dt} = \beta \\
A: \quad (\gamma_1, \theta) \rightarrow (\gamma_2, \theta)
\end{align*}
\]

where \( M = \{(\rho, \theta): \rho = \gamma_1\} \) and \( N = \{(\rho, \theta): \rho = \gamma_2\} \).

We first note that the trajectories of the corresponding differential system are the logarithmic spirals given by

\[
\rho = \rho_0 e^{(\alpha/\beta)(\theta - \theta_0)}, \quad \alpha < 0, \beta > 0
\]

which tend to the stationary position \( \rho = 0 \) as \( \theta \to \infty \).

Since every trajectory of the impulsive differential system starting in the region \( \rho > \gamma_1 \) will hit the circle \( \rho = \gamma_2 \), it is sufficient to consider only the trajectories starting on the circle \( N \). The trajectory \( \rho = \gamma_2 e^{(\alpha/\beta)(\theta - \theta_0)} \) starting at \((\gamma_2, \theta_0)\) hits the circle \( M \) at \((\gamma_1, \theta_1)\) where \( \theta_1 = \theta_0 + \beta t_1 \equiv \theta_0 + \Delta \), \( \Delta = \frac{\beta}{\alpha} \ln \frac{\gamma_1}{\gamma_2} \) and the operator \( A \) switches it to \((\gamma_2, \theta_1)\) on \( N \). Then the motion takes place along the spiral \( \gamma_2 e^{(\alpha/\beta)(\theta - \theta_1)} \) until it meets \( M \) again at \((\gamma_1, \theta_2)\) which gets transferred to the point \((\gamma_2, \theta_2)\) on \( N \), \( \theta_2 = \theta_0 + 2\Delta \). It is easy to see that after the \( n^{th} \) encounter, the motion is at the point \((\gamma_2, \theta_0 + n\Delta)\). Therefore, in order to study the motion, it is sufficient to consider the distribution of all points \( \theta_n = \theta_0 + n\Delta \) on the circle \( N \).

If \( \frac{\Delta}{2\pi} \) is a rational number, that is \( \Delta = 2\pi (p/q) \) where \( p \) and \( q \) are positive integers relatively prime to each other, then

\[
\theta_q = \theta_0 + q\Delta = \theta_0 (\text{mod} \ 2\pi).
\]

Thus the trajectory is closed and the motion is
periodic. Since \( \theta_0 \) is arbitrary, it is clear that all trajectories starting on \( N \) are periodic.

Suppose next that \( \frac{A}{2\pi} \) is not rational. Let us divide the circle \( N \) into \( k \) equal arcs, each subtending a length \( \frac{2\pi}{k} \). Since \( \frac{A}{2\pi} \) is not rational, it never happens that \( \theta_n = \theta_0 \) (mod \( 2\pi \)). Among the first \((k+1)\) points \( \theta_0, \theta_0 + \Delta, \ldots, \theta_0 + k\Delta \) (mod \( 2\pi \)), there are two points on the same arc, say \( \theta_0 + m\Delta, \theta_0 + \ell\Delta, \) \( m > \ell \) and let \( s = m - \ell \). The angle \( s\Delta \) is less than \( \frac{2\pi}{k} \) and in the sequence \( \{ \theta_0 + ns\Delta \) (mod \( 2\pi \)) \}, any two consecutive angles differ by the same number which is less than \( \frac{2\pi}{k} \). Let \( \epsilon > 0 \) be arbitrary and for \( k \) sufficiently large, we can obtain \( \frac{2\pi}{k} < \epsilon \). Now, in any \( \epsilon \)-neighborhood of any point on the circle \( N \), there is at least one element of the set \( \{ \theta_0 + ns\Delta \} \). This shows that the trajectories starting in the region \( x_1^2 + x_2^2 > \gamma_1^2 \) are everywhere dense in the annulus \( x_1^2 + x_2^2 > \gamma_1^2 \) (i.e. the trajectories fill up the annulus densely).

1.2. EXISTENCE AND CONTINUATION.

Let \( \Omega \subset \mathbb{R}^n \) be an open set and let \( D = \mathbb{R}_+ \times \Omega \). Let us suppose that for each \( k = 1, 2, \ldots, \) \( \tau_k \in C[\Omega, (0, \infty)] \), \( \tau_k(x) < \tau_{k+1}(x) \) and \( \lim_{k \to \infty} \tau_k(x) = \infty \) for \( x \in \Omega \). For convenience of notation, we shall introduce \( \tau_0(x) \equiv 0 \) and suppose that \( k \) always runs from 1 to \( \infty \). Also, we recall that \( S_k : t = \tau_k(x) \) are the surfaces.
Consider the initial value problem for impulsive differential system given by

\[
\begin{aligned}
x'(t, x) &= f(t, x), \quad t \neq \tau_k(x), \\
\Delta x &= I_k(x), \quad t = \tau_k(x), \\
x(t_0^+) &= x_0, \quad t_0 \geq 0
\end{aligned}
\]

(1.2.1)

where \( f: D \rightarrow \mathbb{R}^n \) and \( I_k : \Omega \rightarrow \mathbb{R}^n \).

Definition 1.2.1. A function \( x: (t_0, t_0 + a) \rightarrow \mathbb{R}^n \), \( t_0 \geq 0, \ a > 0 \), is said to be a solution of (1.2.1) if

(i) \( x(t_0^+) = x_0 \) and \( (t, x(t)) \in D \) for \( t \in [t_0, t_0 + a] \),

(ii) \( x(t) \) is continuously differentiable and satisfies \( x'(t) = f(t, x(t)) \) for \( t \in [t_0, t_0 + a] \) and \( t \neq \tau_k(x(t)) \),

(iii) if \( t \in [t_0, t_0 + a] \) and \( t = \tau_k(x(t)) \), then \( x(t^+) = x(t) + I_k(x(t)) \), and at such \( t \)'s we always assume that \( x(t) \) is left continuous and \( s \neq \tau_j(x(s)) \) for any \( j \), \( t < s < \delta \), for some \( \delta > 0 \).

We note that, instead of the usual initial condition \( x(t_0) = x_0 \), we have imposed the limiting condition \( x(t_0^+) = x_0 \) which, in general, is natural for systems (1.2.1) since \( (t_0, x_0) \) may be such that \( t_0 = \tau_k(x_0) \) for some \( k \). Whenever \( t_0 \neq \tau_k(x_0) \) for any \( k \), we shall understand the initial condition \( x(t_0^+) = x_0 \) in the usual sense, that is, \( x(t_0) = x_0 \).

Unlike ordinary differential systems, (1.2.1) may not have any solution at all even if \( f \) is continuous (or continuously differentiable) since the only solution \( x(t) \) of the problem \( x' = f(t, x), \ x(t_0) = x_0 \), may totally lie on a surface \( S_k \), say. For example, consider \( x' = 1, \ t \neq \tau_k(x) \) and the rest as in Example 1.1.3. Then it is easy to see that there is no solution through the point \( (1, 1) \).